Solutions to Problem Set from
Electromagnetic Fields and Waves

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Problem (8.1): Prove the BAC–CAB Rule and Derive the Curl Identity

(a) Proof of the BAC–CAB Identity:

Show that for any three vectors **A**, **B**, and **C**,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}). \tag{7.120}$$

Proof: Using index notation with the Levi-Civita symbol ϵ_{ijk} (and Einstein summation over repeated indices), write

$$\big[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})\big]_i = \epsilon_{ijk} A_j (\mathbf{B} \times \mathbf{C})_k = \epsilon_{ijk} A_j \epsilon_{klm} B_l C_m.$$

Using the standard identity

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}, \tag{7.7}$$

we obtain

$$\begin{aligned}\big[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})\big]_i &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j B_l C_m \\ &= B_i (A_j C_j) - C_i (A_j B_j).\end{aligned}$$

In vector notation this is equivalent to

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}).$$

(b) Derivation of the Curl Identity:

Show that

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}. \tag{7.121}$$

Proof: Write the i^{th} component of the double curl:

$$\big[\nabla \times (\nabla \times \mathbf{E})\big]_i = \epsilon_{ijk} \partial_j (\nabla \times \mathbf{E})_k = \epsilon_{ijk} \partial_j \big(\epsilon_{klm} \partial_l E_m\big).$$

Again, using the identity (7.7),

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl},$$

we obtain

$$\begin{aligned}
\big[\nabla \times (\nabla \times \mathbf{E})\big]_i &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\partial_j\partial_l E_m \\
&= \partial_i\partial_j E_j - \partial_j\partial_j E_i \\
&= \partial_i(\nabla \cdot \mathbf{E}) - \nabla^2 E_i.
\end{aligned}$$

In vector form this is exactly

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}.$$

Problem (8.2): Capacitance and Energy of a Parallel-Plate Capacitor

Assume two parallel plates of area A, separated by a distance d, held at a potential difference V. (Neglect fringing fields by assuming the plates are part of an infinite capacitor.)

(a) Capacitance via Gauss' Law:

For an infinite parallel-plate capacitor the electric field between the plates is uniform:

$$E = \frac{V}{d}.$$

By Gauss' Law, the surface charge density is

$$\sigma = \epsilon_0 E = \epsilon_0 \frac{V}{d}.$$

The total charge on one plate is

$$Q = \sigma A = \epsilon_0 \frac{VA}{d}.$$

The capacitance is defined as

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d}.$$

(b) Displacement Current Equals External Current:

In a capacitor, even though no conduction current flows between the plates, a changing electric field produces a displacement current. The displacement current density is

$$J_D = \epsilon_0 \frac{\partial E}{\partial t}.$$

The total displacement current is

$$I_D = \epsilon_0 A \frac{d}{dt}\left(\frac{V}{d}\right) = \frac{\epsilon_0 A}{d}\frac{dV}{dt} = C\,\frac{dV}{dt}.$$

By the continuity equation, this displacement current is equal to the conduction current entering the capacitor.

(c) Stored Energy in Terms of Capacitance:

The energy density in an electric field is

$$u_E = \frac{1}{2}\epsilon_0 E^2.$$

Thus, the total stored energy is

$$U = u_E \cdot (Ad) = \frac{1}{2}\epsilon_0 \left(\frac{V}{d}\right)^2 (Ad) = \frac{1}{2}\frac{\epsilon_0 A}{d}V^2 = \frac{1}{2}CV^2.$$

(d) Energy Storage with a 10 V, 10 A·h Battery:

A battery rated at 10 V and 10 A·h delivers energy:

$$E = V \times (\text{Ampere-hours}) = 10\,\text{V} \times (10\,\text{A} \cdot \text{h}).$$

Since $1\,\text{A} \cdot \text{h} = 3600\,\text{C}$, the total charge is $10 \times 3600 = 36000\,\text{C}$ and the energy is

$$E = 10\,\text{V} \times 36000\,\text{C} = 360\,000\,\text{J}.$$

To store this energy in a capacitor with plate separation $d = 10^{-6}$ m (1 μm) and vacuum dielectric (so $C = \epsilon_0 A/d$), the stored energy is

$$U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\epsilon_0 A}{d}V^2.$$

Set $U = 360\,000\,\text{J}$ and solve for A:

$$A = \frac{2Ud}{\epsilon_0 V^2}.$$

Substitute $U = 3.6 \times 10^5\,\text{J}$, $d = 10^{-6}\,\text{m}$, $\epsilon_0 \approx 8.85 \times 10^{-12}\,\text{F/m}$, and $V = 10\,\text{V}$:

$$A = \frac{2 \times 3.6 \times 10^5 \times 10^{-6}}{8.85 \times 10^{-12} \times 100} \approx \frac{0.72}{8.85 \times 10^{-10}} \approx 8.14 \times 10^8\,\text{m}^2.$$

This enormous area is required.

If each capacitor plate is a square of side $0.1\,\text{m}$ (area $0.01\,\text{m}^2$), the number of plates needed is

$$N = \frac{8.14 \times 10^8}{0.01} = 8.14 \times 10^{10}.$$

If these plates are stacked with a spacing of $10^{-6}\,\text{m}$ each, the total height is

$$H = N \times 10^{-6}\,\text{m} \approx 8.14 \times 10^4\,\text{m} \quad \text{(about 81 km)}.$$

Problem (8.3): Magnetic Field and Energy in a Solenoid

(a) Magnetic Field of an Infinite Solenoid:

For an infinite solenoid with n turns per meter carrying a current I, use Ampère's Law:

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}}.$$

Choosing a rectangular Amperian loop inside the solenoid, the field is uniform and parallel to the axis, so

$$H = nI.$$

In SI units, the magnetic flux density is

$$B = \mu_0 H = \mu_0 nI.$$

(b) Energy Stored in a Solenoid:

The energy density in a magnetic field is

$$u_B = \frac{B^2}{2\mu_0}.$$

For an ideal solenoid (neglecting fringing) with volume $V = \pi r^2 l$, the total stored energy is

$$U = u_B V = \frac{B^2}{2\mu_0}(\pi r^2 l) = \frac{(\mu_0 nI)^2}{2\mu_0}\pi r^2 l = \frac{\mu_0 n^2 I^2}{2}\pi r^2 l.$$

(c) Outward Force on a 10 T MRI Magnet:

For a 10 T magnet with bore diameter 1 m (radius $r = 0.5$ m) and length $l = 2$ m, the magnetic pressure is given by

$$P = \frac{B^2}{2\mu_0}.$$

Using $B = 10\,\mathrm{T}$ and $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H/m}$,

$$P = \frac{100}{2(4\pi \times 10^{-7})} = \frac{100}{8\pi \times 10^{-7}} \approx \frac{100}{2.51 \times 10^{-6}} \approx 4 \times 10^7\,\mathrm{N/m}^2.$$

The force is this pressure times the cross-sectional area of the bore:

$$A = \pi(0.5)^2 \approx 0.785\,\mathrm{m}^2,$$

so

$$F \approx 4 \times 10^7 \times 0.785 \approx 3.14 \times 10^7\,\mathrm{N} \quad (\sim 31\,\mathrm{MN}).$$

Problem (8.4): Definition of the Ampere

The ampere was formerly defined as:

The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of

(a) Verification:

For two long parallel wires separated by $r = 1$ m, the force per unit length is given by

$$\frac{F}{L} = \frac{\mu_0 I^2}{2\pi r}.$$

Substitute $\mu_0 = 4\pi \times 10^{-7}\,\text{H/m}$ and $r = 1$ m:

$$\frac{F}{L} = \frac{4\pi \times 10^{-7}\, I^2}{2\pi} = 2 \times 10^{-7}\, I^2.$$

Setting $F/L = 2 \times 10^{-7}\,\text{N/m}$, we find

$$2 \times 10^{-7}\, I^2 = 2 \times 10^{-7} \quad \Longrightarrow \quad I^2 = 1 \quad \Longrightarrow \quad I = 1\,\text{A}.$$

Thus, a current of 1 A produces the stated force.

(b) Discussion:

Defining the ampere via the force between idealized infinite wires depends on a specific geometry (infinite length, negligible cross-section) and on a force that is not directly fundamental. This definition can be difficult to realize experimentally with high accuracy.

Problem (8.5): The Kibble Balance

The Kibble balance measures mass by equating mechanical and electrical power. It operates in two phases.

(a) Static Phase:

A current I passes through a coil in a spatially inhomogeneous magnetic field with a vertical gradient $\frac{\partial B_z}{\partial z}$. The force on an infinitesimal current element is

$$d\mathbf{F} = I\, d\mathbf{l} \times \mathbf{B}.$$

For the vertical (z) component, integrating over the coil yields

$$F_z = I \left(\frac{\partial B_z}{\partial z} \right) L_{\text{eff}},$$

where L_{eff} is an effective length (a geometrical factor of the coil). In equilibrium, this force balances the gravitational force on the mass mg:

$$I\, L_{\text{eff}} \frac{\partial B_z}{\partial z} = mg.$$

(b) Dynamic Phase:

When the coil moves vertically at constant speed v, a voltage is induced by Faraday's Law:

$$V = -\frac{d\Phi}{dt} = -v\, \frac{d\Phi}{dz},$$

where the flux $\Phi = B_z A_{\text{eff}}$. Hence,

$$V = -v\, A_{\text{eff}} \frac{\partial B_z}{\partial z}.$$

(c) Combining the Results:

From the static phase:

$$mg = I\, L_{\text{eff}} \frac{\partial B_z}{\partial z},$$

and from the dynamic phase:

$$V = v\,A_{\text{eff}}\,\frac{\partial B_z}{\partial z}.$$

Eliminating the gradient (and the effective lengths, assuming they are related by the geometry of the coil), we obtain

$$mg = \frac{I}{v}\,V,$$

or

$$mg = \frac{I\,V}{v}.$$

This equation expresses the mass in terms of the measurable electrical quantities I, V, and the velocity v.

(d) Why Measure Voltage and Current Separately?

The measurements are taken in two distinct phases (static for force and dynamic for induced voltage) because doing so avoids interference between the two effects. If measured simultaneously, the separation between the electrical and mechanical effects would be ambiguous, and systematic errors could result.

Problem (8.6): Estimating Electric Field Strengths from Radiation

(a) Sunlight:

Sunlight at peak delivers a power density $I = 1000\,\text{W/m}^2$. The relation between the intensity and the electric field amplitude is:

$$I = \frac{1}{2}c\epsilon_0 E^2,$$

so

$$E = \sqrt{\frac{2I}{c\epsilon_0}}.$$

Using $c = 3 \times 10^8\,\text{m/s}$ and $\epsilon_0 = 8.85 \times 10^{-12}\,\text{F/m}$:

$$E = \sqrt{\frac{2 \times 1000}{3 \times 10^8 \times 8.85 \times 10^{-12}}} \approx \sqrt{\frac{2000}{2.655 \times 10^{-3}}} \approx \sqrt{7.53 \times 10^5} \approx 868\,\text{V/m}.$$

(b) Focused Laser Beam:

For 1 W of power focused to an area A:

- If $A = 1\,\text{mm}^2 = 10^{-6}\,\text{m}^2$, then the intensity is

  $$I = \frac{1\,\text{W}}{10^{-6}\,\text{m}^2} = 10^6\,\text{W/m}^2.$$

  The electric field amplitude is

  $$E = \sqrt{\frac{2 \times 10^6}{3 \times 10^8 \times 8.85 \times 10^{-12}}} \approx \sqrt{7.53 \times 10^8} \approx 27.5\,\text{kV/m}.$$

- If focused to the diffraction limit, say $A = 1\,\mu\text{m}^2 = 10^{-12}\,\text{m}^2$, then

$$I = \frac{1}{10^{-12}} = 10^{12}\,\text{W/m}^2,$$

and

$$E = \sqrt{\frac{2 \times 10^{12}}{3 \times 10^8 \times 8.85 \times 10^{-12}}} \approx \sqrt{7.53 \times 10^{14}} \approx 8.68 \times 10^7\,\text{V/m},$$

or roughly 87 MV/m.